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The Misconception of Maximum Power and Power Factor in Thermoelectrics

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Abstract

It is commonly claimed that achieving maximum power from a thermoelectric generator necessitates electrical load matching conditions instead of the operating condition derived for maximum generator efficiency. Here we explain why the electrical load matching claim for maximum power in a design optimization is flawed and show that the load condition derived for maximum efficiency always produces more power. Finally, we consider a CPM generator, and prove that the electrical condition for maximum efficiency is indeed the electrical condition for maximum power, maximum power density, maximum power/cost of thermoelectric material, and maximum power/weight of thermoelectric material when the leg length of the thermoelectric generator is a design variable.

Introduction

Since the beginning of the study of thermoelectric devices, the electrical load matching condition, where the resistance of the electrical load (R_L) equals the electrical resistance (R_{TE}) of the thermoelectric generator (TEG), has been utilized in the design of generators. Okhotin (1972) [1] described this as a matter of convenience, beginning with Rayleigh in 1885 until Telkes in 1947 [2, 3]. In 1957, Ioffe explicitly described $R_L/R_{TE} = 1$ as the "maximum power" condition separate from the $R_L/R_{TE} = \sqrt{1 + Z\overline{T}}$ condition for "maximum efficiency" [4] (although this was originally derived in 1909 [5]). Since then, most texts on thermoelectrics describe two separate electrical conditions for design optimizations [1, 4, 6–15].

The separate maximum power and efficiency conditions are valid for a thermoelectric generator that is already constructed (i.e., the geometry is fixed). However, when we discuss the "optimum design" in thermoelectrics, it is almost always meant that the geometry of the thermoelectric module is designed to fit the application, and thus the length of the thermoelectric legs can be varied. Recent full parameter optimizations have contradicted the conventional wisdom by showing that maximum power and maximum efficiency occur at nearly the same electrical operating conditions if either the leg length or ΔT across the generator are allowed to vary [16–19]. In these studies, the load resistances for maximum power and maximum efficiency differ by less than 5%; these differences are likely due to approximations made within the models, or slight errors within the multidimensional optimizations.

Apertet, et al. [20] have identified the electrical operating condition as a primary area of concern within Ref. [21]. Following the conventional wisdom, they argue that maximum efficiency and maximum power occur at different reduced current densities. As such, they argue that the design using the electrical condition for maximum efficiency in Ref. [21] is misdirected. This response is an attempt to explain and dispel the common misconceptions about the operating conditions required for maximum power and maximum efficiency (as previously described in Ref. [22]).

Relative current density and κ_{eff}

Apertet, et al., are correct that the ratio of the heat rate into the TEG (q_h) to the temperature difference across the generator (ΔT) depends on the operating conditions of the generator (specifically electrical current or load resistance). This dependence does allow for the definition of an effective thermal conductivity of the TEG. However, the goal of the effective thermal conductivity derived in Ref. [21] is to define a κ_{eff} such that it does not explicitly depend on the operating conditions (electrical or otherwise) in an optimally designed TEG. Along with being optimally designed, the follow-on assumption is that it is operated at the optimal electrical load condition.

Power and efficiency

In the following, we first discuss the traditional approach to power optimization, and the flaws inherent in this approach. Next, we address the controversy surrounding the optimization for maximum power vs. maximum efficiency in CPM generators. We show that, if the TE leg length is considered to be a design variable (not a fixed value), both maximum power and efficiency can be achieved at the same operating condition. Finally, we discuss an alternative approach to power optimization which does not constrain the TE leg length to a fixed value.

Traditional power optimization

The output power (P) of a TEG depends on the current I and the load resistance R_L as:

$$P = I^2 R_L \tag{1}$$

Within the constant property model (CPM) approximation, and ignoring thermal and electrical contact resistance, the current (I) through the TEG and the load can be expressed in terms of the voltage across the TEG ($V = \alpha\Delta T$) and total resistance:

$$I = \frac{\alpha\Delta T}{R_{TE} + R_L} \tag{2}$$

Combining Eq. (1) and (2) gives:

$$P = \frac{\alpha^2\Delta T^2}{(R_L + R_{TE})^2} R_L \tag{3}$$

Next, we define m as the ratio of the load and TE resistances ($m = R_L/R_{TE}$). The electrical resistance of the TEG ($R_{TE} = \rho l/A$) can be written in terms of the TEG geometry (total area of TE legs A and leg length l) and the material resistivity (ρ). Eq. (3) can then be recast as Eq. (4), the canonical description of power in a CPM generator.

$$P = \frac{\alpha^2\Delta T^2 A}{\rho l}\frac{m}{(1+m)^2} \tag{4}$$

In traditional power optimization, maximizing Eq. (4) is often done piece-wise, with each term maximized separately. This is mathematically incorrect and amounts to performing partial derivatives (rather than a full derivative), in which all other variables are kept constant. For example, optimizing only m leads one to conclude that the maximum power point occurs when $m = 1$, but this conclusion can only be arrived at when all other variables in the equation are fixed. In this highly constrained case (all variables other than m in Eq. (4) fixed), the traditional approach is correct that the maximum power and maximum efficiency occur at different load resistances.

For any specific value of l, the m=1 condition will indeed give more power than $m = \sqrt{1+zT}$. This statement is the basis for the assertion that for all designs, the m=1 condition gives more power, albeit at the expense of more heat. However, when this logic is extrapolated to the limit of infinite heat flux, then infinite power is obtained at l=0. Recognizing that l=0 is unphysical, a minimum value of l is often set, for which the maximum power is achieved at m=1. However, this still amounts to a non-global optimization of the expression for maximum power (Eq. (4)), since l is held constant at a value of $l_{\min}$. A global maximum can be found when l is slightly reduced and the $m = \sqrt{1+zT}$ operating condition is used.

Eq. (4) also contains the power factor (α^2/ρ); this has been used to argue that TE material development should focus on the optimization of the power factor, rather than zT [7–9]. The following analysis should reinforce the requirement that the thermal conductivity κ must also be included in the figure of merit.

Finally, the presence of l in the denominator could lead one to conclude that power is maximized when $l = 0$. Indeed, with the ability to supply an arbitrarily high heat flux (and without losses due to contact resistance), an arbitrarily high power density should result. However, such a solution is nonphysical and mathematically the presence of a variable that could lead to an infinite solution in a maximization problem should not simply be ignored as this amounts to keeping it constant.

Evaluation of the $m = 1$ operating condition

From the above discussion, it is clear that design optimization for maximum power requires additional constraints. Because the validity of additional constraints can be debated, a proof by contradiction is used following [22]: it is conjectured that for a design problem where m and l can be varied, the maximum power solution (including additional constraints such as temperature and heat flow) is found with $m = 1$. Then we shall show below that using the same constraints, there exists a smaller TEG with $m > 1$ that produces more power. Since this contradicts the original conjecture that the maximum power solution has $m = 1$, this conjecture must be false. Thus $m = 1$ is never the electrical condition for maximum power when m and l are design variables.

For a CPM generator, the heat rate into the generator is given by:

$$Q_h = \alpha T_h I + \frac{\kappa A}{l}\Delta T - \frac{1}{2}I^2 R_{TE} \tag{5}$$

Using the definitions of m and R_{TE}, we can rewrite I (Eq. (2)) as:

$$I = \frac{\alpha\Delta T}{(m+1)}\frac{A}{\rho l} \tag{6}$$

This allows us to rewrite the heat rate as:

$$Q_h = \frac{\alpha^2 T_h\Delta T}{m+1}\frac{A}{\rho l} + \frac{\kappa A}{l}\Delta T - \frac{\alpha^2\Delta T^2}{2(m+1)^2}\frac{A}{\rho l} \tag{7}$$

The conjectured maximum power solution with $m = 1$ has design constraints for the values of Q_h, A, ΔT, T_h, and constant material properties. We denote the device with $m = 1$ as TEG-1. The heat rate of TEG-1 can be written as:

$$Q_{h,1} = \frac{\alpha^2 T_h\Delta T A}{2\rho l_1} + \frac{\kappa A}{l_1}\Delta T - \frac{\alpha^2\Delta T^2 A}{8\rho l_1} \tag{8}$$

With the above constraints, the leg length (l_1) is defined from Eq (8).

To test the non-global optimization $m = 1$, we consider another generator with the same operating conditions listed above (A, ΔT, T_h and material properties), except m and l are allowed to vary. This generator (TEG-2) has a leg length l_2 and $m = \sqrt{1+zT_{\text{avg}}}$ (see [22] for explanation of the choice of m value). In this generator, the heat rate is given as:

$$Q_{h,2} = \frac{\alpha^2 T_h\Delta T}{1+\sqrt{1+zT_{\text{avg}}}}\frac{A}{\rho l_2} + \frac{\kappa A}{l_2}\Delta T - \frac{\alpha^2\Delta T^2}{2\left(1+\sqrt{1+zT_{\text{avg}}}\right)^2}\frac{A}{\rho l_2} \tag{9}$$

We now equate the heat rates into TEG-1 and TEG-2 ($Q_{h,1}$=$Q_{h,2}$). Because the operating condition m differs between TEG-1 and TEG-2, the length length l must change to maintain a constant heat rate into the generators. The required value for l_2 in terms of l_1 is:

$$\frac{l_2}{l_1} = \frac{4}{3}\left(\frac{1+2\sqrt{1+zT_{\text{avg}}}}{\left(1+\sqrt{1+zT_{\text{avg}}}\right)^2}\right) \tag{10}$$

Now consider the output power by the two generators using Eq. (4):

$$P_1 = \frac{\alpha^2\Delta T^2 A}{4\rho l_1} \quad P_2 = \frac{\alpha^2\Delta T^2 A}{\rho l_2}\frac{\sqrt{1+zT_{\text{avg}}}}{\left(1+\sqrt{1+zT_{\text{avg}}}\right)^2} \tag{11}$$

Taking the ratio of these two quantities gives:

$$\frac{P_2}{P_1} = \frac{3\sqrt{1+zT_{\text{avg}}}}{1+2\sqrt{1+zT_{\text{avg}}}} \tag{12}$$

If we instead want to consider the volumetric power density, this can be written in terms of Eq.s (10) and (12) (recalling that the areas of the two generators are equal):

$$\frac{P_2/V_2}{P_1/V_1} = \frac{P_2/(Al_2)}{P_1/(Al_1)} = \frac{P_2 l_1}{P_1 l_2} \tag{13}$$

From Eq. (10), l_2 is less than l_1 for all nonzero zT_{avg}, and thus the ratio l_1/l_2 is always greater than unity.

We thus see that $m = 1$ is not an optimum design based on metrics of power (Eq. (12)), power per unit volume (Eq. (13)), or efficiency (since both generators have the same Q_h, power is directly proportional to efficiency).

Optimum electrical conditions

The counterexample ($m = \sqrt{1+zT_{\text{avg}}}$) shows improved performance compared to $m = 1$; the following will demonstrate that this is in fact the optimum m, provided l is an adjustable parameter. We begin by setting $Q_{h,1}$ equal to the general Q_h expression (Eq.s (7) and (8)). Here, the TEG with undefined m and l is again denoted TEG-2. From this heat rate constraint, the leg length ratio l_1/l_2 is given by:

$$\frac{l_1}{l_2} = \frac{(1+m_2)^2\,(8+4zT_h-z\Delta T)}{8\,(1+m_2)\,(1+m_2+zT_h)-4z\Delta T} \tag{14}$$

From Eq. (4), we can determine the ratio of output powers from these two generators:

$$\frac{P_2}{P_1} = \frac{4m_2}{(1+m_2)^2}\frac{l_1}{l_2} \tag{15}$$

Substituting Eq. (14) into Eq. (15) gives:

$$\frac{P_2}{P_1} = \frac{m_2\,(8+4zT_h-z\Delta T)}{2\,(1+m_2)\,(1+m_2+zT_h)-z\Delta T} \tag{16}$$

Taking the derivative of P_2/P_1 in terms of m_2 to determine the maximum of this function gives a value for m_2:

$$m_2 = \sqrt{1+zT_{\text{avg}}} \tag{17}$$

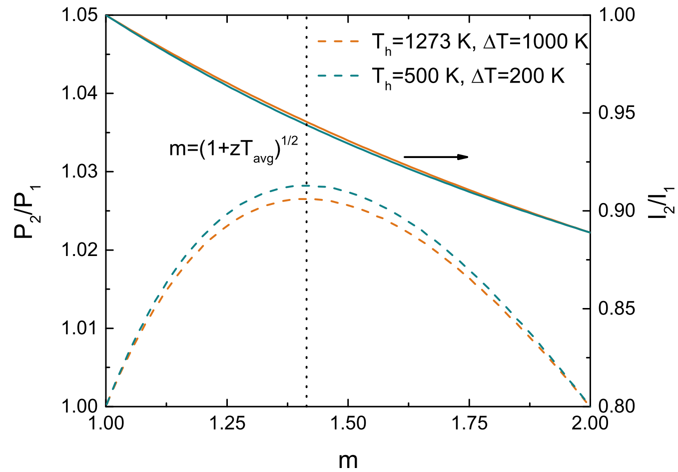

Fig. 1: Power (dashed lines) and leg length (solid lines) ratios as functions of m. Maximum power is achieved at $m = \sqrt{1+zT_{\text{avg}}}$, at which point the leg length is also reduced (calculated for $zT_{\text{avg}} = 1$).

From Fig. 1, it can clearly be seen that power is maximized when $m = \sqrt{1+zT_{\text{avg}}}$, rather than when $m = 1$. Since the heat rate into each generator is the same, power is directly related to efficiency ($P = \eta Q$), and the $m = \sqrt{1+zT_{\text{avg}}}$ also operates at the maximum efficiency. Thus, the long-standing belief that maximum power and maximum efficiency occur at separate load conditions is a product of overly constrained optimization (fixing l). On the right axis of Fig. 1, it can be seen that the leg length of the $m = \sqrt{1+zT_{\text{avg}}}$ generator is shorter than the leg length of the $m = 1$ generator. The optimized generator with $m = \sqrt{1+zT_{\text{avg}}}$ generator is lighter, thinner, and has lower material costs. There is thus no design metric for which an $m = 1$ design is preferred.

We note that, in recent years, this conclusion has been reached independently by several groups by allowing either the leg length or the ΔT across the TEG to vary [16–19].

Alternative consideration of maximum power

While Eq. (4) was effective at demonstrating that maximum efficiency is directly connected to maximum power, an alternative expression has been developed which may be more amenable to seeing this connection. In [21], we demonstrate that the maximum power can be expressed as:

$$P_{max} = \frac{\Delta T_{supply}\eta_{r,d}}{4T_h\Theta_{Hx}} \tag{18}$$

where Θ_{Hx} is the heat exchanger thermal resistance and $\eta_{r,d}$ is the reduced device efficiency. One can directly see the connection between power and efficiency in this expression. Further, this equation lends itself well to the process of designing a generator given a fixed heat source and ΔT_{supply}. As discussed in [21], because the heat exchangers are the physically largest component of the system, these are typically chosen first, which sets Θ_{Hx}. For maximum power, the temperature drop across the TE should be half of the total ΔT_{supply}, which can be used to set T_h. Once a TE material is chosen, then the zT value is known and the reduced device efficiency $\eta_{r,d}$ can be calculated. This allows one to easily estimate the maximum power achievable. Lastly, it is implicit in Eq. (18) that the thermal resistances of the heat exchangers and the TE are equal (see discussion in Ref. [21]). In order to satisfy this condition, κ_{eff} can be used to calculate the leg length required for thermal resistance matching.

Although it is true that the calculation of κ_{eff} and thus the leg length rely on the $u = s$ assumption, any rational generator design will be such that u is as close to s as possible, because this design gives both maximum power and maximum efficiency. The utility of κ_{eff} lies in it's ability to quickly give a close estimate of the design conditions necessary for achieving maximum power.

Thermal Resistance

Apertet, et al. state that defining the ratio of TE to heat exchanger thermal resistances as $\omega = \Theta_{TE}/\Theta_{Hx}$ is confusing because power is maximized for an infinite ω if Θ_{Hx} is allowed to vary. It is true that power is indeed maximized when $\Theta_{Hx} = 0$; however, this situation is highly unphysical as it would be impossible to design a heat exchanger system with zero thermal resistance. Additionally, as discussed in [21], we frame the design problem in terms of a given waste heat source, for which a heat exchanger would be the first system component selected, thus fixing Θ_{Hx}. From Eq. 11 in [21], it is then clear than power is maximized when $\omega = 1$.

Conclusion

Achieving maximum performance (power and efficiency) from a TEG is conceptually non-trivial, as it requires the optimization of the TEG geometry and electrical operating conditions (current and load resistance), as well as the associated heat exchanger design. The temperature dependences of individual TE material properties adds further complexity to this optimization problem. As such, it is not surprising that there is confusion concerning maximum power or efficiency.

Using a CPM generator, we have shown that, when l is considered to be a design variable, maximum power and maximum efficiency occur at the same operating condition. Furthermore, this operating condition (defined by $m = \sqrt{1 + zT_{\text{avg}}}$ for CPM) results in a higher power output than the traditional operating conditions cited in power optimizations ($m = 1$). Additionally the shorter leg length required for this design means that the generator will be lighter, thinner, and have lower material costs than generators designed with $m = 1$.

The $m = \sqrt{1 + zT_{\text{avg}}}$ operating condition also corresponds to the $u \approx s$ condition for a CPM generator. Any rational generator design will optimize the generator for this condition to provide maximum power and efficiency. For this reason, κ_{eff} (derived using the $u = s$ assumption) can be used to simplify this design problem with minimal deviations from actual values. Thus, the expression for κ_{eff} from Ref. [21] does not explicitly depend on the generator operating conditions.

[1] Okhotin, A., Efremov, A., Okhotin, V. & Pushkarskii, A. (Ft. Belvoir Defense Technical Information Center, Charlottesville, VA, USA, 1972).
[2] Rayleigh. *Phil. Mag.* **20**, 361 (1885).
[3] Telkes, M. The Efficiency of Thermoelectric Generators I. *Journal of Applied Physics* **18**, 1116 (1947).
[4] Ioffe, A. (Academic Press, New York, USA, 1960).
[5] Altenkirch, E. *Physikalische Zeitschrift* **10**, 560–580 (1909).
[6] Goldsmid, H. (Methuen, 1960).
[7] Cadoff, I. & Miller, E. (Reinhold Publishing Cooperation, New York, USA, 1960).
[8] Heikes, R. & Ure, R. (Interscience Publishers, 1961).
[9] Sutton, G. (McGraw-Hill, 1966).
[10] Harman, T. & Honig, J. (McGraw-Hill, 1967).
[11] Rosi, F. Thermoelectricity and thermoelectric power generation. *Solid-state Electronics* **11**, 833–868 (1968).
[12] Cobble, M. *Calculations of Generator Performance*, chap. 39 (CRC Press, Boca Raton, FL, USA, 1995).
[13] Goldsmid, H. (Springer, New York, USA, 1995).
[14] Min, G. *Thermoelectric Module Design Theories*, chap. 11 (CRC Press, Taylor & Francis Group, Boca Raton, FL, USA, 2006).
[15] Goldsmid, H. (Springer-Verlag, New York, USA, 2010).
[16] McCarty, R. Thermoelectric Power Generator Design for Maximum Power: It's All About ZT. *Journal of Electronic Materials* **42**, 1504–1508 (2012).
[17] Yazawa, K. & Shakouri, A. Optimization of power and efficiency of thermoelectric devices with asymmetric thermal contacts. *Journal of Applied Physics* **111**, 024509 (2012).
[18] Gomez, M., Reid, R., Ohara, B. & Lee, H. Influence of electrical current variance and thermal resistances on optimum working conditions and geometry for thermoelectric energy harvesting. *Journal of Applied Physics* **113**, 174908 (2013).
[19] Freunek, M., Muller, M., Ungan, T., Walker, W. & Reindl, L.M. New Physical Model for Thermoelectric Generators. *Journal of Electronic Materials* **38**, 1214-1220 (2009).
[20] Apertet, Y., Ouerdane, H., Goupil, C. & Lecoeur, P. Comment on "Effective Thermal Conductivity in Thermoelectric Materials". *Journal of Applied Physics* (2013).
[21] Baranowski, L., Snyder, G. & Toberer, E. Effective Thermal Conductivity in Thermoelectric Materials. *Journal of Applied Physics* **113**, 204904 (2013).

[22] Snyder, G. J. *Thermoelectric Power Generation: Efficiency and Compatibility*, chap. 9 (CRC Press, Taylor & Francis Group, Boca Raton, FL, USA, 2006).